▨ AN EXAMPLE FROM CLASS...

- A total charge Q is uniformly spread out over a flat disk of radius R. What is the electric field @ a point directly above or below the center of the disk?

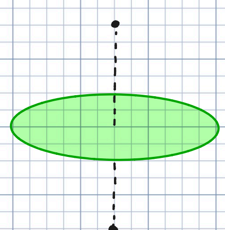
- Spreading charge over a disk gives a surface charge density, so we need to evaluate:

$$\vec{E}(\underbrace{\vec{r}}_{\substack{\text{The point where}\\ \text{we wish to know}\\ \text{the electric field.}}}) = \frac{1}{4\pi\varepsilon_0}\int_{\underbrace{S}_{\substack{\text{Visit every pt.}\\ \text{on the disk...}}}} \overbrace{da'\,\sigma(\vec{r}\,')\,\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}}$$

... and add up the electric field produced @ $\vec{r}$ by the infinitesimal bit of charge $dq(\vec{r}\,') = da'\,\sigma(\vec{r}\,')$ found @ $\vec{r}\,'$!

- In this case the charge Q is spread uniformly over the surface, and the area of a disk is πR^2, so the surface charge density will be constant: $\sigma = Q/\pi R^2$.

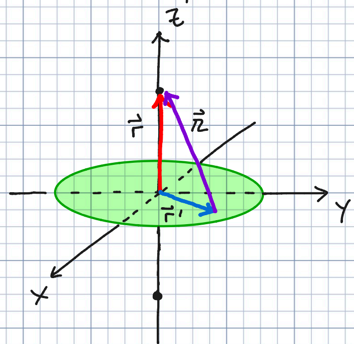
- First, how do we describe the disk and the point where we want to know $\vec{E}$?



Always try to describe things in the simplest manner possible. The description should take symmetry into account! In this case, points above & below the center lie on an axis of symmetry.

It makes sense to use Cylindrical Polar Coordinates here. The z-axis can be the axis of symmetry, and it's easy to describe the disk.

- We'll put the origin @ the center of the disk. Then:



Disk: $z = 0, \quad 0 \le s' \le R, \quad 0 \le \phi' < 2\pi$

Pt. on disk: $\vec{r}\,' = s'\hat{s} + 0\,\hat{z}$

$= s'\cos\phi'\,\hat{x} + s'\sin\phi'\,\hat{y} + 0\,\hat{z}$

Pt. above/below center of disk: $\vec{r} = 0\hat{x} + 0\hat{y} + z\hat{z}$

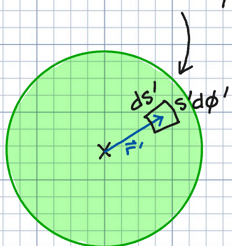
Sep. vector: $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

$= -s'\cos\phi'\,\hat{x} - s'\sin\phi'\,\hat{y} + z\,\hat{z}$

$$|\vec{\mathscr{r}}| = \sqrt{\underbrace{s'^2\cos^2\phi' + s'^2\sin^2\phi'}_{s'^2} + z^2}$$

$$= \sqrt{s'^2 + z^2}$$

$da' = s'd\phi'ds'$



NOTE: We used Cartesian unit vectors in $\vec{\mathscr{r}}$, since $\hat{s}$ depends on one of the coords we'll integrate over (ϕ').

- Before jumping right into the integral, work out the integrand:

$$da'\,\sigma(\vec{r}\,')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = d\phi'ds's'\frac{Q}{\pi R^2}\frac{(-s'\cos\phi'\,\hat{x} - s'\sin\phi'\,\hat{y} + z\,\hat{z})}{(s'^2 + z^2)^{3/2}}$$

$$= d\phi'ds'\frac{Q}{\pi R^2}\frac{(-s'^2\cos\phi'\,\hat{x} - s'^2\sin\phi'\,\hat{y} + s'z\,\hat{z})}{(s'^2 + z^2)^{3/2}}$$

- Putting this all together:

$$\vec{E}\underbrace{(0,0,z)}_{\text{Pt. above/below center of disk}} = \frac{1}{4\pi\varepsilon_0}\underbrace{\frac{Q}{\pi R^2}}_{\sigma \text{ is constant here}}\underbrace{\int_0^{2\pi}d\phi'\int_0^R ds'}_{\text{Visit every point}}\frac{(-s'^2\cos\phi'\,\hat{x} - s'^2\sin\phi'\,\hat{y} + s'z\,\hat{z})}{(s'^2 + z^2)^{3/2}}$$

- Now, we could just evaluate this. But it's worth asking what we *expect* the answer to be.

- Based on the symmetry of the charge distribution (it is spread evenly over the disk) and the point where we're evaluating $\vec{E}$, we expect to get an electric field pointing straight up or down along that symmetry axis. So $\vec{E}$ should have a z-component, but no x- or y-component.

- Can we see this in our integral? Look @ the x-component:

$$E_x(0,0,z) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\pi R^2}\int_0^R ds' \int_0^{2\pi} d\phi' \frac{(-s'^2\cos\phi')}{(s'^2+z^2)^{3/2}}$$

The only ϕ'-dependence in the integrand is a factor of $\cos\phi'$, which gets integrated over a full period: $0 \to 2\pi$.

$$\int_0^{2\pi} d\phi' \cos\phi' = 0$$

$$\hookrightarrow E_x(0,0,z) = 0$$

- The same is true for E_y. So we have:

$$\vec{E}(0,0,z) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\pi R^2}\hat{z}\int_0^{2\pi} d\phi' \int_0^R ds' \frac{s' z}{(s'^2+z^2)^{3/2}}$$

- There's no ϕ' dependence, so the integral over ϕ' just gives 2π:

$$\vec{E}(0,0,z) = \frac{1}{2\varepsilon_0}\frac{Q}{\pi R^2} z\,\hat{z}\int_0^R ds' \frac{s'}{(s'^2+z^2)^{3/2}}$$

- To evaluate the integral over s', we make a change-of-variable:

$$u = s'^2 + z^2 \rightarrow du = 2s'ds' \rightarrow ds's' = \frac{1}{2}du$$

$$s'=0 \rightarrow u = z^2 \quad \& \quad s'=R \rightarrow u = R^2+z^2$$

$$\int_0^R ds' \frac{s'}{(s'^2+z^2)^{3/2}} = \int_{z^2}^{R^2+z^2} du \, \frac{1}{2} \frac{1}{u^{3/2}} = \frac{1}{2}\left(-\frac{2}{u^{1/2}}\right)\Bigg|_{z^2}^{R^2+z^2}$$

$$= -\left(\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{\sqrt{z^2}}\right)$$

$$= \frac{1}{\sqrt{z^2}} - \frac{1}{\sqrt{R^2+z^2}}$$

- We have to be careful here. Our integrand was explicitly <u>positive</u>: s' is b/t 0 & R, & $z^2 > 0$, so $s'/(s'^2+z^2)^{3/2} \geq 0$. And if we add up lots of non-negative numbers, the result must <u>also</u> be non-negative. But z could be positive (above the center) or negative (below the center), so we <u>must</u> write the result of the s' integral as

$$\int_0^R ds' \frac{s'}{(s'^2+z^2)^{3/2}} = \frac{1}{|z|} - \frac{1}{\sqrt{R^2+z^2}}$$

- So our final answer is:

$$\frac{z}{|z|} = \begin{cases} 1, & z>0 \\ -1, & z<0 \end{cases}$$

$$\vec{E}(0,0,z) = \frac{1}{2\varepsilon_0}\frac{Q}{\pi R^2}\left(\frac{z}{|z|} - \frac{z}{\sqrt{R^2+z^2}}\right)\hat{z}$$

If $Q>0$, this points up for $z>0$ & down for $z<0$, as expected.

- Does our answer make sense? The direction seems right, but how else can we check our work?

- Imagine you were above the disk ($z>0$) & you moved very, very far away. Far compared to what? Since the only other length here is the radius of the disk, "very far away" must mean $z \gg R$. Then:

$$\vec{E}(0,0,z) = \frac{1}{2\varepsilon_0}\frac{Q}{\pi R^2}\left(1 - \frac{\cancel{z}}{\cancel{z}\sqrt{1+\frac{R^2}{z^2}}}\right)$$

$$= \frac{1}{2\varepsilon_0}\frac{Q}{\pi R^2}\left(1 - \underbrace{\frac{1}{\sqrt{1+R^2/z^2}}}\right)$$

TAYLOR: $(1+x^2)^{-\frac{1}{2}} \simeq 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 + \ldots$

- So when $z \gg R$, which means R^2/z^2 is very small, $\vec{E}$ @ the point $(0,0,z)$ is approximately

Powers of $\frac{R}{z}$ that are really small when $z \gg R$.

$$\vec{E}(0,0,z) = \frac{1}{2\varepsilon_0}\frac{Q}{\pi R^2}\left(\cancel{1} - \cancel{1} + \frac{1}{2}\frac{R^2}{z^2} - \frac{3}{8}\frac{R^4}{z^4} + \ldots\right)\hat{z}$$

$$= \frac{1}{2\varepsilon_0}\frac{Q}{\pi \cancel{R^2}}\,\frac{1}{2}\,\frac{\cancel{R^2}}{z^2}\,\hat{z} + \ldots$$

$$= \frac{1}{4\pi\varepsilon_0}\,Q\,\frac{\hat{z}}{z^2} + \ldots$$

Electric field @ $(0,0,z)$ due to a pt. charge Q @ $(0,0,0)$.

- The further away we move, the smaller quantities like R^4/z^4 become compared to R^2/z^2, and the more $\vec{E}$ looks like the electric field for a point charge Q. And that makes sense: when we're far away ($z \gg R$) the disk does look like a point, so its electric field <u>should</u> look approximately like the electric field for a point charge.

- What else can we learn from our result? What if, instead of moving very far away, we zoomed in very close to the disk so that $z << R$? Then:

powers $\left(\frac{z}{R}\right)^4$ & higher

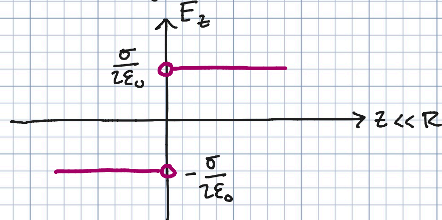
$$\frac{z}{\sqrt{R^2+z^2}} = \frac{z}{R\sqrt{1+z^2/R^2}} = \frac{z}{R} \times \left(1 - \frac{1}{2}\frac{z^2}{R^2} + \dots\right)$$

$$\hookrightarrow \vec{E}(0,0,z<<R) = \frac{1}{2\varepsilon_0}\frac{Q}{\pi R^2}\left(\frac{z}{|z|} - \frac{z}{R} + \frac{1}{2}\frac{z^3}{R^3} + \dots\right)\hat{z}$$

- If we were <u>really</u> close — say, $z/R = 10^{-6}$ or any other value much less than 1 — this is:

$$\vec{E}(0,0,z<<R) \simeq \frac{\sigma}{2\varepsilon_0}\frac{z}{|z|}\hat{z} \qquad \sigma = \frac{Q}{\pi R^2}$$

- The factor $z/|z|$ is 1 above the disk $(z>0)$ and -1 below the disk $(z<0)$. So, just fractionally above the disk $\vec{E} \simeq (\sigma/2\varepsilon_0)\hat{z}$, while just below it is $\vec{E} \simeq -(\sigma/2\varepsilon_0)\hat{z}$. It is approximately constant very close to the disk, and has the same magnitude on both sides, but it <u>abruptly</u> flips direction when we go from one side to the other!



- This is our first encounter with an important property of electric fields produced by charge distributions: they can be <u>discontinuous</u> at points where there is a surface charge density.